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# On the topology of the space of representations

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## Abstract

We give a proof that the set of discrete and faithful  $SL_2$  representations of the fundamental group of a topologically finite Riemann surface is closed in the set of all  $SL_2$  representations. This result is well known.

## 1 Introduction

If we treat Teichmüller space as Fricke moduli, we first study the set of discrete and faithful  $SL_2(\mathbf{R})$  or  $SL_2(\mathbf{C})$  representations of the fundamental group of a topologically finite Riemann surface. And making it as a topological space we consider it as a subspace of all  $SL_2(\mathbf{R})$  or  $SL_2(\mathbf{C})$  representations of the fundamental group of a topologically finite Riemann surface. Then it is well known (for example [MS]) that it is a closed subset. The key idea to prove this result is using so called Jørgensen's inequalities [J]. In this article we show a proof of this result in the elementary style. It should be remarked that if we consider the set of discrete and faithful representations as a subset of the more restricted space of representations, we can also show the openness of it for the case of  $SL_2(\mathbf{R})$  representations by using the rigidity of the finite sided fundamental domains (for example [M]. for compact case see [W]). From this topological properties Teichmüller space of a topologically finite Riemann surface has the structure of a semi algebraic set.

## 2 Preliminaries

In this paper  $\Gamma$  means a non abelian free group of finite rank or surface group.

**Assertion 1**  $\Gamma$  is torsion free.

*Proof.*

Because of the uniformization theorem,  $\Gamma$  can be considered as a discrete subgroup of  $PSL_2(\mathbf{R})$  the analytic automorphism group of the upper half plane  $H$  and  $\Gamma$  acts on  $H$  fixed point freely. Hence any non identity element of  $\Gamma$  is hyperbolic or parabolic element of  $PSL_2(\mathbf{R})$  and this shows that  $\Gamma$  is torsion free.

**Assertion 2** For any  $\alpha, \beta \in \Gamma$  with  $\alpha\beta \neq \beta\alpha$ , put  $G := \langle \alpha, \beta \rangle$  the subgroup of  $\Gamma$  generated by  $\alpha$  and  $\beta$ . Then  $G$  is a free group of rank two.

*Proof.*

As in the proof of Assertion 1, we may assume that  $\Gamma$  is a discrete subgroup of  $PSL_2(\mathbf{R})$  and acts on  $H$  fixed point freely. Then the quotient space  $G \backslash H$  has the structure of a Riemann surface and its first homology group  $H_1(G \backslash H, \mathbf{Z})$  is a quotient group of  $\mathbf{Z}^2$ . If  $G \backslash H$  is compact, then  $H_1(G \backslash H, \mathbf{Z})$  is isomorphic to  $\mathbf{Z}^{2g}$  where  $g (\geq 2)$  is the genus of this surface. Hence  $G \backslash H$  is an open Riemann surface and its fundamental group  $G$  is a free group. Because  $G$  is generated by two elements which are not commutative, it is a free group of rank two.

**Assertion 3** The center of  $\Gamma$  is trivial.

*Proof.*

Let  $\gamma$  be a center of  $\Gamma$ . If there exists a hyperbolic element  $\alpha$  of  $\Gamma$  (where we take some realization  $\Gamma \subset PSL_2(\mathbf{R})$ ), we may assume by conjugation that the representative of  $\alpha$  in  $SL_2(\mathbf{R})$  is a diagonal matrix. Then the assumption  $\gamma\alpha = \alpha\gamma$  implies that  $\gamma$  has also a diagonal matrix as its representative. Since  $\Gamma$  is non abelian, there exists  $\beta \in \Gamma$  whose representative is not diagonal and  $\gamma\beta = \beta\gamma$  shows that  $\gamma$  must be an identity. Similar argument holds for the case that  $\alpha$  is parabolic.

### 3 Results

**Theorem 1** A representation  $\rho : \Gamma \rightarrow SL_2(\mathbf{C})$  is discrete (i.e.  $\rho(\Gamma) \subset SL_2(\mathbf{C})$  is a discrete subgroup) and faithful (i.e.  $\rho$  is injective) if and only if the following inequalities hold:

for any  $\alpha, \beta \in \Gamma$  with  $\alpha\beta \neq \beta\alpha$ , put  $A := \rho(\alpha)$  and  $B := \rho(\beta)$ . Then  $A, B \in SL_2(\mathbb{C})$  satisfy so called Jørgensen's inequality

$$|tr^2 A - 4| + |tr[A, B] - 2| \geq 1.$$

Proof.

(if) We assume that  $\rho$  is not faithful. Then there exists  $\alpha (\neq id.) \in \Gamma$  such that  $A = \rho(\alpha) = E$  (identity matrix). As the center of  $\Gamma$  is trivial by Assertion 3, there exists  $\beta \in \Gamma$  with  $\alpha\beta \neq \beta\alpha$  and put  $B = \rho(\beta)$ . Then

$$|tr^2 A - 4| + |tr[A, B] - 2| = 0$$

a contradiction. Hence  $\rho$  is faithful. Next suppose that  $\rho$  is faithful but not discrete. Then there exists a sequence  $(X_n)_{n \geq 1} \subset \rho(\Gamma)$  with  $X_n \rightarrow E$  in  $SL_2(\mathbb{C})$ . Then for any  $B \in \rho(\Gamma)$ ,  $|tr^2 X_n - 4| \rightarrow 0$  and  $|tr[X_n, B] - 2| \rightarrow 0$ . Therefore there exists  $N(B) \in \mathbb{N}$  depending only on  $B$  such that for  $n \geq N(B)$

$$|tr^2 X_n - 4| + |tr[X_n, B] - 2| < 1.$$

Hence if there exists  $n \geq N(B)$  such that  $X_n B \neq B X_n$  then because of the faithfulness of  $\rho$ , we put  $A = X_n$  and get a contradiction of the assumption. Therefore we may assume in the following that  $X_n B = B X_n$  for all  $n \geq N(B)$ . If  $B$  is not parabolic, we may suppose that  $B$  is a diagonal matrix by conjugation. Then  $X_n$  is also diagonal. By Assertion 3, there exists  $C \in \rho(\Gamma)$  such that  $CB \neq BC$  in other words  $C$  is not diagonal, hence  $CX_n \neq X_n C$  for any  $n \geq N(B)$ . Then there exists  $N(C) \in \mathbb{N}$  depending only on  $C$  such that for  $n \geq N(C)$

$$|tr^2 X_n - 4| + |tr[X_n, C] - 2| < 1.$$

and  $CX_n \neq X_n C$  which contradicts the assumption. Similar argument holds for the case that  $B$  is parabolic and we conclude that  $\rho$  is discrete and faithful.

(only if) Because  $\rho$  is faithful and by Assertion 2,  $\langle A, B \rangle$  is a free subgroup of  $SL_2(\mathbb{C})$ . Assume that

$$|tr^2 A - 4| + |tr[A, B] - 2| < 1$$

and put  $B_0 := B, B_{n+1} := B_n A B_n^{-1}$  ( $n = 0, 1, 2, \dots$ ). Then the completely same proof of Lemma 1 of [J] shows that  $B_{n+1}$  converges to  $A$  in  $SL_2(\mathbb{C})$  but the discreteness of  $\rho$  means that  $\langle A, B \rangle$  is a discrete subgroup of  $SL_2(\mathbb{C})$  hence  $B_{n+1} = A$  for sufficiently large  $n \in \mathbb{N}$ . But this contradicts that  $\langle A, B \rangle$  is a free group of rank two.

**Corollary 1** *A representation  $\rho : \Gamma \rightarrow SL_2(\mathbf{R})$  is discrete and faithful if and only if the following inequalities hold:*

*for any  $\alpha, \beta \in \Gamma$  with  $\alpha\beta \neq \beta\alpha$ , put  $A := \rho(\alpha)$  and  $B := \rho(\beta)$ . Then  $A, B \in SL_2(\mathbf{R})$  satisfy so called Jørgensen's inequality*

$$|tr^2 A - 4| + |tr[A, B] - 2| \geq 1.$$

**Proof.**

(if) We can use the same argument in the proof of the above Theorem to show the faithfulness of  $\rho$ . Then  $\rho(\Gamma)$  is a non abelian subgroup of  $SL_2(\mathbf{R})$ , we can find a hyperbolic or parabolic element of  $\rho(\Gamma)$  and we can prove the discreteness of  $\rho$  in the same way.

(only if) The natural inclusion  $\mathbf{R} \subset \mathbf{C}$  induces the discrete and faithful representation

$$\rho : \Gamma \rightarrow SL_2(\mathbf{R}) \subset SL_2(\mathbf{C}).$$

**Corollary 2** *The set of discrete and faithful  $SL_2(\mathbf{C})$  (resp.  $SL_2(\mathbf{R})$ ) representations of the fundamental group of a Riemann surface of topologically finite type is closed in the set of all  $SL_2(\mathbf{C})$  (resp.  $SL_2(\mathbf{R})$ ) representations.*

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